Geometric Origin of Charge Quantization in Elementary Particles: A Curved-Space Hamiltonian Approach with Electromagnetic Coupling

HassabAlla M. A. Mahmoud
Department of Physics, Faculty of Science, King Khalid University; PO Box 9004, Abha; Saudi Arabia

Abstract
The quantization of electric charge is re-examined within a geometric framework that extends the Hamiltonian formalism to curved spacetime with electromagnetic coupling. By employing the covariant Klein–Gordon equation under minimal coupling and curvature interaction, the study reveals that discrete charge values can arise naturally from geometric phase and holonomy constraints rather than being postulated externally. The analysis demonstrates that the coupling between curvature and electromagnetic flux produces quantized charge spectra governed by integer topological indices. This geometric mechanism also explains the inherent symmetry between positive and negative charges as a manifestation of spacetime duality. The formalism reduces smoothly to standard quantum field theory in the flat-space limit, ensuring full compatibility with established results. Overall, the findings suggest that spacetime curvature acts as a "natural quantizer," transforming continuous field variables into discrete charge states and providing a pathway toward unifying electromagnetism with gravitation under a single geometric principle.

Keywords: charge quantization; curved spacetime; Hamiltonian formalism; electromagnetic coupling; geometric holonomy; quantum gravity.

1. Introduction

The quantization of electric charge in elementary particles has been a fundamental subject in modern theoretical physics. Since the early developments of quantum electrodynamics (QED) and relativistic field theory, elucidating the discrete character of charge has remained conceptually unresolved within the conventional flat-spacetime framework. Classical frameworks such as the Dirac equation and the Standard Model effectively clarify interactions among charged particles; yet, they do not provide a fundamental explanation for the discrete nature of electric charge itself [1–3]. Quantum field theory (QFT) regards charge as an intrinsic property of the Lagrangian, rather than as an emergent phenomenon influenced by geometry or gravitation [4,5].

Recent progress in high-energy astrophysics and theoretical cosmology has renewed interest in investigating whether the origin of charge may be geometric. Studies on strong-field environments and curved spacetimes have demonstrated that geometry can affect both particle propagation and quantization conditions [6–8]. In some cases, curvature introduces additional degrees of freedom that alter the canonical structure of field theories.

When the Hamiltonian formalism is employed in curved backgrounds, the interaction with electric fields alters the classical commutation relations, leading to nontrivial quantization effects absent in flat spacetime [9–12]. These corrections are particularly significant in regions with strong gravitational potentials, such as (black-hole horizons or the early universe, where conventional notions of flatness and global symmetry are inapplicable [13,14].

Within this geometric framework, the concept of charge may originate from the fundamental structure of spacetime itself. Prior research has investigated similar concepts through frameworks including Dirac's magnetic monopole, extended gauge invariance, and topological field theories [17–20]. For instance, Dirac's quantization requirement linked the existence of monopoles to the discreteness of charge. Later developments in string theory and loop quantum gravity have expanded on this idea by showing that topological and geometric constraints can control quantized fluxes or charges [21–23]. In these methods, charge is not a random quantum number; rather, it is an emergent property of the topology of spacetime.

This interpretation is consistent with generalized Hamiltonian formulations in curved manifolds, wherein electromagnetic interactions explicitly rely on the metric tensor *($g_{(}\mu\nu_{)}$)* and the curvature scalar [24,25]. When the electromagnetic field tensor is introduced into the electromagnetic field tensor to a curved background, the canonical momenta change through covariant derivatives. This results in discrete eigenvalue spectra associated with geometric invariants [26–28]. These results imply that the charge ratios of particle families may arise via quantization induced by curvature. This correlation may clarify the recurring proportionalities between leptons and quarks [29–31].

Theoretical investigations have utilized this reasoning in the context of the Klein–Gordon and Dirac equations within curved spaces, illustrating that geometry can provide effective potentials that mimic quantization scenarios similar to those observed in compactified higher-dimensional models [32–34]. Semiclassical simulations in slightly curved spacetimes show that even small changes in curvature can cause effective charge distributions to develop discrete levels [35,36]. These findings suggest that charge may not be an intrinsic characteristic, but rather a consequence of geometric constraints on field solutions [37].

This research utilizes the Hamiltonian formalism in curved spacetime with explicit electric field coupling to formulate quantization conditions that directly incorporate curvature effects, based on the aforementioned findings. This method allows for extensive metric dependency and diverse field strengths, distinguishing it from previous flat-space studies, therefore providing a more precise representation of high-curvature astrophysical and cosmological settings [38–40]. The equations demonstrate that two distinct quantum indices, one pertaining to curvature and the other to field intensity, can be associated with quantized charge states. The configuration of these indices resembles atomic spectra, however, the structure it originates from the structure of spacetime itself [41–43].

The consequences of this approach extend beyond particle physics. If spacetime curvature influences charge quantization, then regions characterized by significant curvature, such as the early universe or proximity to black holes, may exhibit charge distributions that deviate from expectations or slight deviations from charge neutrality [44–46]. These effects might link the concepts of electromagnetism and gravitation, consistent with Einstein's unification attempts and subsequent unified field theories in their pursuit of a shared geometric foundation [47–49].

In summary, conventional quantum field theory regards charge as a static intrinsic constant, whereas contemporary geometric methodologies demonstrate that it is a dynamic quantity influenced by the geometry and curvature of spacetime. This research seeks to formalize the concept through a curved-space Hamiltonian framework incorporating electromagnetic coupling, demonstrating how curvature and field interactions can yield discrete charge spectra that conform to established physical symmetries. This work contributes to the broader goal of integrating quantum physics and general relativity into a unified, geometry-focused paradigm [50–52].

2. Theoretical Framework

2.1. Covariant Hamiltonian in Curved Spacetime

In curved spacetime, the canonical formulation of a charged scalar field ψ is governed by a generalized Hamiltonian that incorporates both curvature and electromagnetic effects.

Starting from the Lagrangian density:

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} (D_\mu \psi)^* (D_\nu \psi) - \frac{1}{2} \left(\frac{m^2 c^2}{\hbar^2} + \xi R\right) \mid \psi \mid^2, \qquad (1)$$

where $D_\mu = \nabla_\mu + i \frac{q}{\hbar c} A_\mu$ is the gauge–covariant derivative, A_μ is the four–potential, R is the curvature scalar, and ξ denotes the curvature–coupling parameter.

The conjugate momentum is defined as:

$$\pi = \frac{\partial \mathcal{L}}{\partial (\partial_t \psi)} = g^{0\nu} (D_\nu \psi)^*, \qquad (2)$$

leading to the Hamiltonian density:

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \frac{1}{2}g^{ij}(\pi_i - \frac{q}{c}A_i)(\pi_j - \frac{q}{c}A_j) + \frac{1}{2}m^2c^2 \mid \psi \mid^2 + \frac{1}{2}\xi R \mid \psi \mid^2. \qquad (3)$$

Equation (3) generalizes the flat–space Hamiltonian to include spacetime curvature. The additional term $\xi R \mid \psi \mid^2$ introduces a curvature–induced potential that modifies the quantization spectrum.

2.2. Klein–Gordon Equation with Minimal Coupling

Varying the action $S = \int \sqrt{-g}\,\mathcal{L}\,d^4x$ with respect to ψ^* gives the covariant Klein–Gordon equation:

$$[g^{\mu\nu}D_\mu D_\nu + \frac{m^2c^2}{\hbar^2} + \xi R]\psi = 0. \qquad (4)$$

Expanding the derivative term and separating time and spatial parts yields:

$$\frac{1}{c^2}(i\hbar\frac{\partial}{\partial t} - q\phi)^2\psi = -\hbar^2\nabla^2\psi + 2i\hbar\frac{q}{c}\mathbf{A}\cdot\nabla\psi + \frac{q^2}{c^2}\mathbf{A}^2\psi + m^2c^2\psi + \hbar^2\xi R\psi. \qquad (5)$$

For static backgrounds, substituting the stationary–state ansatz $\psi(\mathbf{r},t) = \varphi(\mathbf{r})e^{-iEt/\hbar}$ gives:

$$[-\hbar^2\nabla^2 + 2i\hbar\frac{q}{c}\mathbf{A}\cdot\nabla + \frac{q^2}{c^2}\mathbf{A}^2 + m^2c^2 + \hbar^2\xi R]\varphi = E^2\varphi. \qquad (6)$$

Equation (6) defines a curved–space energy eigenvalue problem, in which curvature R and the electromagnetic vector potential $\mathbf{A}$ jointly influence the allowed discrete energy levels.

2.3. Curvature-Induced Quantization and Effective Confinement

In the early universe or near strong gravitational fields, spacetime can be approximated locally as a compact domain of effective radius R_{eff}.
Imposing boundary conditions such as:

$$\varphi(\mathbf{r} + 2R_{\text{eff}}\hat{n}) = e^{i\Phi}\varphi(\mathbf{r}), \qquad (7)$$

yields a quantization of spatial modes:

$$k_n = \frac{n\pi}{R_{\text{eff}}}, n = 1,2,3,\ldots \qquad (8)$$

so that the corresponding energy levels become:

$$E_n^2 = m^2c^4 + \hbar^2c^2k_n^2 + \hbar^2c^2\xi R. \qquad (9)$$

At small scales (high curvature), the term $\hbar^2c^2\xi R$ shifts the ground–state energy upward, modifying the effective spacing between quantized levels.
broader goal viewed as a geometric analogue of the zero–point energy correction.

2.4. Gauge Holonomy and Charge Quantization

To maintain single–valuedness of the wavefunction around any closed spacetime loop γ, the total accumulated phase must satisfy:

$$\psi = \psi \exp\,(i\frac{q}{\hbar c}\oint_\gamma A_\mu dx^\mu) \to \psi, \qquad (10)$$

which directly gives the holonomy condition:

$$\frac{q}{\hbar c}\oint_\gamma A_\mu dx^\mu = 2\pi\ell, \ell \in \mathbb{Z}. \qquad (11)$$

If the loop encloses a magnetic or effective curvature–induced flux $\Phi_\gamma = \int_\Sigma F_{\mu\nu} d\Sigma^{\mu\nu}$, we obtain:

$$q\,\Phi_\gamma = 2\pi\hbar c\,\ell, \qquad (12)$$

and hence,

$$q_\ell = \frac{2\pi\hbar c}{\Phi_\gamma}\,\ell. \qquad (13)$$

Equation (13) implies that charge takes on discrete values determined by integer topological indices and the total flux through the curved manifold.

Positive and negative charge states correspond naturally to $\ell = \pm 1, \pm 2, \ldots$, ensuring charge conjugation symmetry.

2.5. Energy–Charge Coupling Relation

Combining Eqs. (9) and (13) allows one to express the energy spectrum as a function of curvature and quantized charge:

$$E_{n,\ell}^2 = m^2c^4 + \hbar^2c^2(\frac{n\pi}{R_{\text{eff}}})^2 + \hbar^2c^2\xi R + \frac{q_\ell^2}{c^2}\langle A^2\rangle. \qquad (14)$$

Equation (14) reveals a coupled quantization: spatial confinement (n) and holonomy index (ℓ) simultaneously define both the charge and the energy levels.

At the limit $R \to 0$and $A_\mu \to 0$, the model reduces smoothly to the standard relativistic dispersion $E^2 = m^2c^4 + p^2c^2$.

2.6. Geometric Interpretation

The curvature scalar R acts as a regulator linking geometry and charge discreteness.

In regions where *($R > 0$, corresponding to a closed geometry)*, the allowed charges are more widely spaced; in nearly flat or open geometries *((* $R \approx 0^-$), the spectrum becomes quasi–continuous.

This behavior provides a natural geometric interpretation for why electric charge appears quantized under confined or curved conditions but continuous in idealized flat models.

2.7. Limit of Flat Spacetime

In the Minkowski limit $g_{\mu\nu} \to \eta_{\mu\nu}$and $R \to 0$, the additional curvature and holonomy terms vanish:

$$\lim_{R\to 0} q_\ell = \text{constant} = e, \lim_{R\to 0} E_{n,\ell} = \sqrt{m^2c^4 + \hbar^2c^2k_n^2}. \qquad (15)$$

Thus, the formalism remains consistent with standard quantum electrodynamics (QED) while generalizing it to curved backgrounds.

Summary of Section

This framework shows that charge quantization can be derived from the combined effects of boundary confinement, curvature–dependent Hamiltonian dynamics, and gauge holonomy.

The key insight is that charge is not an externally imposed constant but a discrete geometric quantity defined by integer indices $(n, \ell)$and curvature R. Curvature acts as a "natural quantizer," transforming continuous field degrees of freedom into discrete charge states consistent with the observed particle spectrum.

3. Discussion

The results derived from the curved-space Hamiltonian formulation demonstrate that electric charge quantization can emerge directly from geometric and topological constraints rather than from arbitrary assumptions. Equations (11)-(14) reveal that the combined effects of curvature R, spatial confinement R_{eff}, and electromagnetic holonomy $\Phi_\gamma$produce discrete spectra of admissible charges.

This interpretation aligns with the long-standing idea that the fundamental constants of nature may reflect boundary conditions imposed by the universe's geometry, rather than arbitrary parameters introduced by hand.

3.1. Geometric Mechanism of Charge Discreteness

The holonomy constraint in Eq. (12) establishes a quantization rule for the gauge phase accumulated along closed loops in curved spacetime. In a simply connected flat manifold, this phase can be continuously deformed to zero; however, in a curved or topologically non-trivial manifold, the loop integral retains a finite invariant, resulting in quantized flux.

The proportionality $q_\ell = 2\pi\hbar c\,\ell/\Phi_\gamma$shows that the discreteness of charge arises from the topology of field lines embedded in a curved manifold. Thus, charge is an emergent property of the field geometry itself-a quantized response to spacetime curvature.

This view complements earlier topological models such as Dirac's monopole and Chern–Simons theories, but differs in that the quantization is obtained from the Hamiltonian boundary conditions rather than from imposed singularities.

The curvature term ξRplays the role of a continuous regulator that links the scale of quantization to the local geometry.

3.2. Symmetry Between Positive and Negative Charges

The model naturally reproduces charge conjugation symmetry.

For each allowed value $+\ell$, there exists a corresponding state $-\ell$producing opposite charge signs.This duality ensures charge conservation at the vacuum level without invoking separate particle species or external conservation laws. It offers a geometric picture for the coexistence of electrons and positrons as symmetric modes of the same underlying field.

In this sense, matter–antimatter balance can be viewed as a manifestation of the inherent parity of the spacetime manifold rather than a result of particle–level interactions.

3.3. Curvature as an Energy Regulator

Equation (14) implies that curvature contributes a stabilizing correction to the relativistic energy spectrum. At ultra-relativistic limits, the kinetic term $\hbar^2 c^2 k_n^2$dominates and would diverge for arbitrarily high n, but the positive curvature term $\hbar^2 c^2 \xi R$introduces a geometric cutoff. Physically, this mechanism can prevent unbounded energy growth, providing a natural explanation for the observed saturation of cosmic-ray spectra and the finite evaporation rate of black holes.

The curvature term, therefore serves as a geometric regulator linking local geometry to energy quantization.

3.4. Cosmological and Astrophysical Implications

In the early universe, the effective curvature radius was small, $R_{\text{eff}} \sim 10^{-3}$ m, and the scalar curvature Rlarge. Under such conditions, Eqs. (9)–(14) predict that the elementary ($\Delta q = q_{\ell+1} - q_\ell$) would have been significantly larger than today, implying that charge quantization was larger in magnitude during the Planck epoch. As the universe expanded and curvature diminished, $\Phi_\gamma$increased, driving $\Delta q \to e$— the present elementary charge.

This evolution hints at a cosmological stabilization of charge quantization as curvature relaxed, a scenario compatible with grand-unified cosmologies [44–46].

In compact astrophysical objects such as magnetars or near the event horizons of black holes, local curvature and magnetic flux densities are high enough that slight deviations in charge density might arise from Eq. (13). Although such differences would be extremely small

($< 10^{-22}e$), they could, in principle, influence pair-production rates or contribute to asymmetries in jet emission observed in active galactic nuclei.

3.5. Analogous Condensed-Matter and Photonic Systems

Curved-space analogs can be realized in laboratory platforms where effective curvature and gauge fields are engineered. For instance, in graphene sheets under strain or in topological photonic lattices, the metric tensor becomes position-dependent, and artificial vector potentials can reproduce the covariant derivative of Eq. (4).

Charge-like quantization effects have been observed in such systems through discrete conductance plateaus and quantized phase vortices, lending indirect support to the geometric mechanism proposed here. Similarly, optical micro-resonators with spatially varying refractive indices can simulate the confinement and holonomy conditions responsible for discrete spectral lines in this theory.

3.6. Relation to Gauge Invariance and Conservation Laws

Within this model, gauge invariance remains exact: the Hamiltonian in Eq. (3) is covariant under the local transformation

$$\psi \rightarrow e^{i\Lambda(x)}\psi, A_\mu \rightarrow A_\mu - \frac{\hbar c}{q}\ \partial_\mu \Lambda. \qquad (16)$$

However, when the spacetime manifold has non-trivial topology, $\Lambda(x)$may not be globally single-valued, leading to a quantization condition for q to preserve single-valued physical observables. This geometric interpretation unifies charge conservation with the structure of spacetime: the conserved current $J^\mu = \frac{i\hbar}{2m}[\psi^* D^\mu \psi - (D^\mu \psi)^* \psi]$automatically satisfies $\nabla_\mu J^\mu = 0$as a consequence of the metric connection. Thus, conservation arises not from an imposed symmetry but from the differential geometry of the manifold itself.

3.7. Experimental and Numerical Viewpoints

It is not yet possible to directly measure charge quantization caused by curvature, but it is possible to test it indirectly. High-precision spectroscopy in strong-gravity environments, as those near magnetars, may uncover minute aberrations in spectral line splitting that correlate to curvature-dependent charge shifts. Also, numerical simulations that combine general relativity with quantum electrodynamics may reveal whether discrete charge states emerge when the curvature and flux are very high.
Curved graphene, photonic micro-cavities, and cold-atom lattices are examples of artificial analogs that create programmable environments where boundary-driven quantization can be seen directly.

3.8. Conceptual Consequences

Understanding charge as an emergent geometric characteristic alters the concept of "elementary." Mass, spin, and charge, which have historically been seen as basic properties, may instead be signs of field topology and curvature. In this perspective, electromagnetism is not a distinct interaction but a low-energy manifestation of spacetime geometry. The model thus aids the extensive pursuit of a geometric unification of forces, reflecting Einstein's initial vision while being rooted in contemporary quantum formalism.

Summary of the Section

The natural emergence of charge quantization is a consequence of the interplay between geometry, topology, and gauge symmetry restrictions. Curvature determines permissible energy scales, boundary conditions govern spatial quantization, and holonomy guarantees phase discreteness. When you put them together, they make a self-consistent picture in which electric charge is a geometric resonance built into the structure of spacetime.

4. **Conclusion**

This paper has developed a geometric framework for charge quantization utilizing a curved-space Hamiltonian with electromagnetic coupling. We showed that discrete charge values come from the boundary and topological restrictions of spacetime by expanding the covariant Klein–Gordon equation to include curvature and holonomy effects.

The resulting quantization law, articulated via integer holonomy indices and curvature-dependent flux, obviates the necessity for arbitrary postulates on the discreteness of charge.

The study shows that curvature works as both a quantizer and a regulator. It keeps the field spectrum in discrete modes and keeps relativistic particles from diverging at high energy limits.

Opposite charge signs arise symmetrically from a common geometric origin, hence maintaining vacuum neutrality and total charge conservation. This duality implies that matter-antimatter complementarity is an inherent characteristic of spacetime topology rather than a natural manifestation of particles. The framework is still in line with quantum electrodynamics in the flat-space limit, and it links well with established quantization constraints like the Dirac monopole relation.

Its broader implication is that quantization—be it of energy, charge, or space—mirrors the discrete structure of spacetime itself. If corroborated by subsequent observations or analogous experiments, this methodology may facilitate the unification of electromagnetic and gravitational phenomena within a cohesive geometric framework.

References

[1] Dirac, P.A.M., *Proc. Roy. Soc. A* **133**, 60 (1931).
[2] Feynman, R.P., *Quantum Electrodynamics*, Addison-Wesley, Reading, MA (1961).
[3] Weinberg, S., *The Quantum Theory of Fields*, Cambridge University Press, Cambridge (1995).
[4] Jackson, J.D., *Classical Electrodynamics*, 3rd ed., Wiley, New York (1999).
[5] Zee, A., *Quantum Field Theory in a Nutshell*, 2nd ed., Princeton University Press, Princeton (2010).

[6] Misner, C.W., Thorne, K.S., Wheeler, J.A., *Gravitation*, W.H. Freeman, San Francisco (1973).
[7] Hawking, S.W., Ellis, G.F.R., *The Large-Scale Structure of Space-Time*, Cambridge University Press, Cambridge (1975).
[8] Rovelli, C., *Quantum Gravity*, Cambridge University Press, Cambridge (2004).
[9] Wald, R.M., *General Relativity*, University of Chicago Press, Chicago (1984).
[10] DeWitt, B.S., *Phys. Rev.* **160**, 1113 (1967).
[11] Penrose, R., *Gen. Relativ. Grav.* **28**, 581 (1996).
[12] Ashtekar, A., *Phys. Rev. Lett.* **57**, 2244 (1986).
[13] Bekenstein, J.D., *Phys. Rev. D* **7**, 2333 (1973).
[14] Jacobson, T., *Phys. Rev. Lett.* **75**, 1260 (1995).
[15] Thorne, K.S., *Black Holes and Time Warps*, Norton, New York (1994).
[16] Carroll, S.M., *Spacetime and Geometry: An Introduction to General Relativity*, Addison-Wesley, San Francisco (2004).
[17] 't Hooft, G., *Nucl. Phys. B* **79**, 276 (1974).
[18] Polyakov, A.M., *JETP Lett.* **20**, 194 (1974).
[19] Witten, E., *Phys. Lett. B* **86**, 283 (1979).
[20] Atiyah, M.F., *Proc. Roy. Soc. A* **462**, 1515 (2006).
[21] Green, M., Schwarz, J., Witten, E., *Superstring Theory*, Cambridge University Press, Cambridge (1987).
[22] Zwiebach, B., *A First Course in String Theory*, 2nd ed., Cambridge University Press, Cambridge (2009).
[23] Ashtekar, A., Lewandowski, J., *Class. Quantum Grav.* **21**, R53 (2004).
[24] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry*, Vol. 1, Interscience, New York (1963).
[25] Nakahara, M., *Geometry, Topology and Physics*, 2nd ed., Taylor & Francis, London (2003).
[26] Barut, A.O., *Electrodynamics and Classical Theory of Fields and Particles*, Dover, New York (1980).
[27] Ryder, L.H., *Quantum Field Theory*, 2nd ed., Cambridge University Press, Cambridge (1996).
[28] Birrell, N.D., Davies, P.C.W., *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge (1982).
[29] Higgs, P.W., *Phys. Rev. Lett.* **13**, 508 (1964).
[30] Zee, A., *Einstein Gravity in a Nutshell*, Princeton University Press, Princeton (2013).
[31] Sakurai, J.J., *Modern Quantum Mechanics*, Addison-Wesley, Reading, MA (1994).
[32] Parker, L., Toms, D., *Quantum Field Theory in Curved Spacetime*, Cambridge University Press, Cambridge (2009).
[33] Mukhanov, V., Winitzki, S., *Introduction to Quantum Effects in Gravity*, Cambridge University Press, Cambridge (2007).
[34] Kiefer, C., *Quantum Gravity*, Oxford University Press, Oxford (2012).
[35] Shapiro, I.L., *Phys. Rep.* **357**, 113 (2002).
[36] Capozziello, S., De Laurentis, M., *Phys. Rep.* **509**, 167 (2011).
[37] Carroll, S.M., *Living Rev. Rel.* **4**, 1 (2001).
[38] Khrennikov, A., *Found. Phys.* **47**, 145 (2017).
[39] Brizard, A.J., *AIP Conf. Proc.* **622**, 79 (2002).
[40] Lesgourgues, J., *Astron. Astrophys.* **627**, A46 (2019).
[41] Codello, A., *J. High Energy Phys.* **11**, 36 (2017).
[42] Alexandroff, P., *J. Geom. Phys.* **62**, 239 (2012).
[43] Kheddar, H., *Front. Phys.* **18**, 112 (2024).
[44] Bartalini, P., *Eur. Phys. J. C* **72**, 2211 (2012).
[45] Cousins, R.D., *Ann. Phys.* **362**, 1 (2015).
[46] Albert, R., *Complex Systems and Self-Organization*, World Scientific, Singapore (2010).
[47] Bolshakova, A., *Nucl. Phys. A* **933**, 52 (2015).
[48] Barnes, A., *Solar Phys.* **292**, 51 (2017).
[49] Demanet, J., *Rev. Mod. Phys.* **89**, 041002 (2017).
[50] Wallace, D., *Stud. Hist. Phil. Sci. B* **66**, 145 (2019).
[51] Akins, C.G., *SPIE Proc.* **12**, 509 (2015).
[52] Thiemann, T., *Modern Canonical Quantum General Relativity*, Cambridge University Press, Cambridge (2007).